

Forward and Converse Theorems of Polynomial Approximation for Exponential Weights on $[-1, 1]$, II

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We consider exponential weights of the form $w := e^{-Q}$ on $[-1, 1]$ where $Q(x)$

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$$Q(x) := \exp_k((1-x)^{-\alpha}), \quad k \geq 0, \alpha > 0,$$

where \exp_k denotes the k th iterated exponential and $\exp_0(x) = x$. We prove converse theorems of polynomial approximation in weighted L_p spaces with norm $\|fw\|_{L_p[-1, 1]}$ for all $0 < p \leq \infty$, to match the forward theorems proved in part I of this paper. © 1997 Academic Press

1. STATEMENT OF RESULTS

There is a well developed theory of weighted polynomial approximation for weights $w: (-1, 1) \rightarrow (0, \infty)$ that behave like Jacobi weights near ± 1 [9]. However, for weights that decay rapidly near ± 1 , this theory does not apply. In this paper, we prove converse theorems of polynomial approximation for even weights

$$w := e^{-Q} \tag{1.1}$$

where $Q: (-1, 1) \rightarrow \mathbb{R}$ is even and grows at least as fast as $(1-x^2)^{-\delta}$, $\delta > 0$, near ± 1 . In part I of this paper [16], we proved Jackson theorems for these weights; that is, we estimated

$$E_n[f]_{w,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)w\|_{L_p[-1, 1]}, \tag{1.2}$$

$0 < p \leq \infty$, where \mathcal{P}_n denote the polynomials of degree at most n .

Our methods are similar to those in [8], where Jackson theorems were proved for Freud weights, and to the follow up papers [2, 3] where Erdős

weights were treated. To state our results, we need to define our class of weights, as well as various quantities. First, we say that a function $f: (a, b) \rightarrow (0, \infty)$ is *quasi-increasing* if $\exists C > 0$ such that

$$a < x < y < b \Rightarrow f(x) \leq Cf(y).$$

DEFINITION 1.1. Let $w := e^{-Q}$, where

(a) $Q: (-1, 1) \rightarrow \mathbb{R}$ is even, continuous, has limit ∞ at 1, and Q' is positive in $(0, 1)$.

(b) $xQ'(x)$ is strictly increasing in $(0, 1)$ with right limit 0 at 0.

(c) The function

$$T(x) := \frac{Q'(x)}{Q(x)} \quad (1.3)$$

is quasi-increasing in $(C, 1)$ for some $0 < C < 1$.

(d) $\exists C_1, C_2, C_3 > 0$ such that

$$\frac{Q'(y)}{Q'(x)} \leq C_1 \left(\frac{Q(y)}{Q(x)} \right)^{C_2}, \quad y \geq x \geq C_3. \quad (1.4)$$

(e) For some $\delta > 0$, $0 < C < 1$, $(1 - x^2)^{1+\delta} Q'(x)$ is increasing in $(C, 1)$. Then we write $w = e^{-Q} \in \mathcal{E}$.

The archetypal example of $w \in \mathcal{E}$ is

$$w(x) := w_{k, \alpha}(x) := \exp(-\exp_k([1 - x^2]^{-\alpha})), \quad k \geq 0, \quad \alpha > 0, \quad (1.5)$$

where $\exp_k = \exp(\exp(\dots))$ denotes the k th iterated exponential and $\exp_0(x) = x$ [16].

We need the condition that $xQ'(x)$ be strictly increasing to guarantee the existence of the *Mhaskar–Rahmanov–Saff number* a_u , the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}, \quad u > 0. \quad (1.6)$$

For those to whom a_u is new, its significance lies partly in the identity [19–21]

$$\|Pw\|_{L_\infty[-1, 1]} = \|Pw\|_{L_\infty[-a_n, a_n]}, \quad P \in \mathcal{P}_n \quad (1.7)$$

and the fact that a_n is the “smallest” such number.

Our modulus of continuity involves two parts, a “main part” and a “tail.” The main part involves r th symmetric differences over the interval $[-a_{1/(2t)}, a_{1/(2t)}]$, and the tail involves an error of weighted polynomial approximation over the remainder of $(-1, 1)$. For $h > 0$, an interval J , and $r \geq 1$, we define the r th symmetric difference

$$\Delta_h^r(f, x, J) := \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right), \quad (1.8)$$

provided all arguments of f lie in J , and 0 otherwise. Sometimes, we just write $\Delta_h^r f(x)$ if it is clear which interval J we are using. Sometimes the increment h will depend on x and the function

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{a_{1/t}}\right|} + T(a_{1/t})^{-1/2}, \quad x \in (-1, 1). \quad (1.9)$$

This is the case in our modulus of continuity

$$\begin{aligned} \omega_{r,p}(f, w, t) := & \sup_{0 < h \leq t} \|w \Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2t)})} \\ & + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P)w\|_{L_p(a_{1/(4t)} \leq |x| \leq 1)} \end{aligned} \quad (1.10)$$

and its averaged cousin

$$\begin{aligned} \bar{\omega}_{r,p}(f, w, t) := & \left[\frac{1}{t} \int_0^t \|w \Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2t)})}^p dh \right]^{1/p} \\ & + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P)w\|_{L_p(a_{1/(4t)} \leq |x| \leq 1)}. \end{aligned} \quad (1.11)$$

(If $p = \infty$, we set $\bar{\omega}_{r,p} := \omega_{r,p}$.) See [16] for further discussion of the modulus. Here we simply note that the function $\Phi_t(x)$ describes the improvement in the degree of approximation near $\pm a_n$, in much the same way that $\sqrt{1-x^2}$ does for Jacobi weights on $[-1, 1]$. The main result of part I of this paper [16] is

THEOREM 1.2. *Let $w := e^{-Q} \in \mathcal{E}$. Let $r \geq 1$ and $0 < p \leq \infty$. Then for $f: (-1, 1) \rightarrow \mathbb{R}$ for which $fw \in L_p(-1, 1)$ (and for $p = \infty$, we require f to be continuous and fw to vanish at ± 1), we have for $n \geq C_3$*

$$E_n[f]_{w,p} \leq C_1 \bar{\omega}_{r,p}\left(f, w, \frac{C_2}{n}\right) \leq C_1 \omega_{r,p}\left(f, w, \frac{C_2}{n}\right), \quad (1.12)$$

where $C_j, j = 1, 2, 3$, do not depend on f or n . Moreover,

$$E_n[f]_{w,p} \leq C_1 \inf_{\rho \in [3/4, 1]} \bar{\omega}_{r,p} \left(f, w, C_2 \frac{\rho}{n} \right). \quad (1.13)$$

(The inequality (1.13) was stated as Theorem 6.3 in [16].)

In establishing converse theorems of polynomial approximation, the K -functional plays a crucial role [9]. In our context, a suitable K -functional is

$$K_{r,p}(f, w, t^r) := \inf_g \{ \|(f - g) w\|_{L_p[-1, 1]} + t^r \|g^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \}, \quad (1.14)$$

where the inf is taken over all g whose $(r-1)$ st derivative is locally absolutely continuous. The presence of the function Φ_t reflects “endpoint influences.” Unfortunately, the K -functional is useful only for $p \geq 1$ as it often vanishes identically for $p < 1$ [6]. So several authors have used the realisation functional, which works for all $0 < p \leq \infty$, though it is not as elegant as the K -functional [3, 6, 8, 12]. In our context, a suitable realisation functional is

$$\bar{K}_{r,p}(f, w, t^r) := \inf_{\deg(P) \leq 1/t} \{ \|(f - P) w\|_{L_p[-1, 1]} + t^r \|P^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \}. \quad (1.15)$$

Our first result concerns the equivalence between the realisation functional and the modulus of continuity:

THEOREM 1.3. *Let $w := e^{-Q} \in \mathcal{E}$. Let $r \geq 1$, $M > 0$ and $0 < p \leq \infty$. Assume, moreover, that there is a Markov–Bernstein inequality of the form*

$$\|P'_n w \Phi_{1/n}\|_{L_p[-1, 1]} \leq Cn \|P_n w\|_{L_p[-1, 1]}, \quad P_n \in \mathcal{P}_n, \quad (1.16)$$

where $C \neq C(n, P_n)$. Then there exists $t_0 > 0$ and $C_j > 0, j = 1, 2, 3$, independent of f, t such that for $t \in (0, t_0)$,

$$C_1 \omega_{r,p}(f, w, t) \leq \bar{K}_{r,p}(f, w, t^r) \leq C_2 \omega_{r,p}(f, w, t) \quad (1.17)$$

and

$$C_1 \bar{\omega}_{r,p}(f, w, t) \leq \bar{K}_{r,p}(f, w, t^r) \leq C_2 \bar{\omega}_{r,p}(f, w, t). \quad (1.18)$$

Moreover,

$$\omega_{r,p}(f, w, Mt) \leq C_3 \omega_{r,p}(f, w, t). \quad (1.19)$$

For $p = \infty$, the Markov–Bernstein inequality (1.16) was established in [18] under additional conditions on Q , namely, conditions on Q'' , which are certainly satisfied for $w_{k,\alpha}$ of (1.5). The methods of [15, 17] and the Christoffel function estimates of [14] enable one to extend (1.16) to all $0 < p \leq \infty$ under the conditions of [18]. Moreover, we are certain that (1.16) is true under scarcely more than we need for $w \in \mathcal{E}$.

Note too that for $p \geq 1$, the methods of [9] should enable one to avoid assuming the Markov–Bernstein inequality for Theorem 1.3. However, one needs to assume this inequality for the corollaries below, so we do not devote any attention to this point.

The inequality (1.19) allows us to simplify the Jackson theorem:

COROLLARY 1.4. *Under the hypotheses of Theorem 1.3, for $n \geq C_2$*

$$E_n[f]_{w,p} \leq C_1 \bar{\omega}_{r,p} \left(f, w, \frac{1}{n} \right) \leq C_1 \omega_{r,p} \left(f, w, \frac{1}{n} \right), \quad (1.20)$$

where C_1, C_2 do not depend on f or n .

From Theorem 1.3 we can also deduce converse theorems of approximation. For the statement, we need $\langle x \rangle$, the greatest integer $\leq x$.

COROLLARY 1.5. *Assume the hypotheses of Theorem 1.3. Let $q := \min\{1, p\}$. There exists t_0 depending only on w, r, p such that for $t \in (0, t_0)$, and $l := \langle \log_2(1/t) \rangle$, we have*

$$\omega_{r,p}(f, w, t)^q \leq C t^{rq} \sum_{j=-1}^l (l-j+1)^{rq/2} 2^{jq} E_{2^j}[f]_{w,p}^q, \quad (1.21)$$

where $C \neq C(f, w, t)$ and we set $E_{2^{-1}} := E_0$.

One can deduce from (1.20) and (1.21) that for $0 < \alpha < r$,

$$\omega_{r,p}(f, w, t) = O(t^\alpha) \Leftrightarrow E_n[f]_{w,p} = O(n^{-\alpha}). \quad (1.22)$$

Finally, we note that for $p \geq 1$, the modulus may be estimated in terms of $f^{(r)}$, and the K -functional and realisation functional are equivalent:

COROLLARY 1.6. *Let $1 \leq p \leq \infty$. Let $w \in \mathcal{E}$.*

(a) *If $f^{(r)}w \in L_p[-1, 1]$, we have*

$$\omega_{r,p}(f, w, t) \leq C_1 t^r \|f^{(r)}w \Phi_t^r\|_{L_p[-1, 1]} \quad (1.23)$$

for $t \in (0, C_2)$. Here $C_j \neq C_j(f, t)$, $j = 1, 2$.

(b) *If also the Markov–Bernstein inequality (1.16) holds, then*

$$1 \leq \bar{K}_{r,p}(f, w, t')/K_{r,p}(f, w, t') \leq C_3, \quad (1.24)$$

for $t \in (0, C_4)$. Here $C_j \neq C_j(f, t)$, $j = 3, 4$.

The paper is organised as follows: In Section 2, we present some technical details related to Q , a_u , and so on. In Section 3, we present a crucial inequality that is proved using ideas of Z. Ditzian, following his proof in [8]. In Section 4, we prove Theorem 1.3. In Section 5, we prove Corollaries 1.4 to 1.6.

We close this section with more notation. Throughout, C, C_1, C_2, \dots denote positive constants independent of n, x , and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L . Moreover, when dealing with, for example, $x, y \in (C, 1)$, it is taken as understood that $C < 1$. The notation $c_n \sim d_n$ means that $C_1 \leq c_n/d_n \leq C_2$ for the relevant range of n . Similar notation is used for sequences of functions. $\langle x \rangle$ denotes the greatest integer $\leq x$. In the sequel, we assume that $w = e^{-Q} \in \mathcal{E}$.

2. TECHNICAL LEMMAS

We begin by recalling some technical results from the first part of this paper [16]. Throughout we assume that $w \in \mathcal{E}$. Recall that Φ_t is defined by (1.9).

LEMMA 2.1. (a) *For $u \geq C$,*

$$Q'(a_u) \sim uT(a_u)^{1/2}. \quad (2.1)$$

Given fixed $\beta > 0$, we have for large u

$$T(a_{\beta u}) \sim T(a_u). \quad (2.2)$$

Given fixed $\alpha > 1$,

$$\frac{a_{\alpha u}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \quad (2.3)$$

For some $\delta > 0$, $C_3 > 0$,

$$T(a_u) \leq C_3 u^{2(1-\delta)}. \quad (2.4)$$

(b) There exists C_1 , such that for $0 < s \leq C_1$, and $s/2 \leq t \leq s$,

$$\Phi_s(x) \sim \Phi_t(x), \quad x \in [-1, 1]. \quad (2.5)$$

Moreover, for $0 < s < t \leq C$,

$$\sup_{x \in [-1, 1]} \frac{\Phi_s(x)}{\Phi_t(x)} \leq C_2 \sqrt{\log \left(2 + \frac{t}{s} \right)}. \quad (2.6)$$

(c) Let $L > 0$. For $t \in (0, t_0)$, $|x|, |y| \leq a_{1/t}$ such that

$$|x - y| \leq Lt\Phi_t(x), \quad (2.7)$$

we have

$$w(x) \sim w(y); \quad \Phi_t(x) \sim \Phi_t(y). \quad (2.8)$$

Proof. (a) This is part of Lemma 2.2 in [16].

(b) This is Lemmas 3.1(b) and 7.1(b) in [16].

(c) This is Lemma 3.2(b) in [16], with a trivial modification to the proof there. ■

Next, we present a restricted range inequality involving a suitable modification of $\Phi_{1/n}$:

LEMMA 2.2. Let $0 < p \leq \infty$, $s > 1$. For $n \geq 1$, define

$$\Psi_n(x) := \left(1 - \left[\frac{x}{a_n} \right]^2 \right)^2 + T(a_n)^{-2}. \quad (2.9)$$

Then for some $n \geq 1$, $0 \leq l \leq n$ and $P \in \mathcal{P}_n$,

$$\|Pw\Psi_n^{l/4}\|_{L_p(-1, 1)} \leq C_1 \|Pw\Psi_n^{l/4}\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}. \quad (2.10)$$

Moreover,

$$\|Pw\Psi_n^{l/4}\|_{L_p(1 \geq |x| \geq a_n)} \leq C_1 e^{-C_2 n T(a_n)^{-1/2}} \|Pw\Psi_n^{l/4}\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}. \quad (2.11)$$

Here $C_j \neq C_j(n, P_n, l)$, $j = 1, 2$. The same result holds for a fixed l (with constants depending on l) if we replace $\Psi_n^{1/4}$ by $\Phi_{1/n}$.

Remark. Note that (2.4) shows that for some $C_3 > 0$, and large enough n ,

$$nT(a_n)^{-1/2} \geq n^{C_3}. \quad (2.12)$$

Proof. For $l=0$, this is Lemma 2.3 in [16]. Next note that

$$T(a_n)^{-2} \leq \Psi_n(x) \leq C \quad \text{in } [-1, 1].$$

Write $l=4j+k$, $0 \leq k \leq 3$. Then as $P\Psi_n^j$ is a polynomial of degree at most $n+l \leq 2n$, the case $l=0$ of (2.11) gives

$$\begin{aligned} \|Pw\Psi_n^{l/4}\|_{L_p(1 \geq |x| \geq a_{s(l+n)})} &= \|Pw\Psi_n^j \Psi_n^{k/4}\|_{L_p(1 \geq |x| \geq a_{s(l+n)})} \\ &\leq C_1 \|Pw\Psi_n^j\|_{L_p(1 \geq |x| \geq a_{s(l+n)})} \\ &\leq C_1 e^{-C_2 n T(a_n)^{-1/2}} \|Pw\Psi_n^j\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}. \end{aligned}$$

Our lower bound for Ψ_n allows us to continue this as

$$\leq C_1 T(a_n)^{k/2} e^{-C_2 n T(a_n)^{-1/2}} \|Pw\Psi_n^{j+k/4}\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}$$

and since (2.12) holds, we deduce (2.11) and hence (2.10). Since $\Psi_n^{1/4} \sim \Phi_{1/n}$ uniformly in $[-1, 1]$ and in n , we also obtain (2.10), (2.11) for a fixed l , with $\Psi_n^{1/4}$ replaced by $\Phi_{1/n}$. ■

We shall need an extension of the Markov–Bernstein inequality (1.16):

LEMMA 2.3. *Let $0 < p \leq \infty$ and Ψ_n be defined by (2.9) for $n \geq 1$. Then for $n \geq 1$, $0 \leq l \leq n$, $P_n \in \mathcal{P}_n$,*

$$\|P_n^{(l+1)} w \Psi_n^{(l+1)/4}\|_{L_p[-1, 1]} \leq C_1 \{n + l T(a_n)^{1/2}\} \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-1, 1]} \quad (2.13)$$

$$\leq C_2 n(l+1) \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-1, 1]}, \quad (2.14)$$

where $C_j \neq C_j(n, l, P_n)$, $j=1, 2$. The same result holds for a fixed l (with constants depending on l) if we replace $\Psi_n^{1/4}$ by $\Phi_{1/n}$.

Proof. It suffices to prove the result for large n . We first construct suitable polynomial approximations of $\Psi_n^{1/4}$. To do this, we use the Christoffel functions $\lambda_m(u, x)$ associated with the ultraspherical weight

$$u(x) := (1 - x^2)^{-3/4}, \quad x \in (-1, 1).$$

It is known [24, p. 36] that $\lambda_m(u, x)^{-1}$ is a polynomial of degree $2m-2$ such that uniformly in m and $x \in (-1, 1)$,

$$\lambda_m(u, x)^{-1} \sim m[1 - x^2 + m^{-2}]^{1/4} \quad (2.15)$$

and given $A > 0$, for $|x| \leq 1 - Am^{-2}$,

$$|\lambda'_m(u, x)/\lambda_m(u, x)| \leq C[1 - x^2 + m^{-2}]^{-1}. \quad (2.16)$$

We choose $m := m(n)$ to be the greatest integer $\leq T(a_n)^{1/2}$ and set

$$R_n(x) := \left[\frac{1}{m} \lambda_m^{-1} \left(u, \frac{x}{a_{3n}} \right) \right]^2.$$

Then R_n has degree $o(n)$ by (2.4). Moreover, (2.15) and (2.16) give

$$R_n(x) \sim \Phi_{1/n}(x) \sim \Psi_n(x)^{1/4}, \quad x \in [-a_{3n}, a_{3n}] \quad (2.17)$$

and

$$|R'_n(x)/R_n(x)| \leq C \Psi_n(x)^{-1/2}, \quad x \in [-a_{3n}, a_{3n}]. \quad (2.18)$$

Next, write $l = 4j + k$, $0 \leq k \leq 3$. Let $Q_1 := P_n^{(l)}$. Let $1 < s < \frac{3}{2}$. From Lemma 2.2,

$$\begin{aligned} \|P_n^{(l+1)} w \Psi_n^{(l+1)/4}\|_{L_p[-1, 1]} &\leq C_1 \|Q'_1 w \Psi_n^{(l+1)/4}\|_{L_p[-a_{s(l+1+n)}, a_{s(l+1+n)}]} \\ &\leq C_2 \|Q'_1 \Psi_n^j R_n^k w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\ &\leq C_3 \{ \|(Q_1 \Psi_n^j R_n^k)' w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\ &\quad + j \|Q_1 \Psi_n^{j-1} \Psi_n' R_n^k w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\ &\quad + k \|Q_1 \Psi_n^j R_n^{k-1} R_n' w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \} \\ &=: C_3(T_1 + T_2 + T_3). \end{aligned} \quad (2.19)$$

Here by (1.16), applied to $Q_1 \Psi_n^j R_n^k$, which has degree at most $n + l + o(n) \leq \frac{5}{2}n$, and as

$$\Psi_n^{1/4} \sim \Phi_{1/(3n)} \quad \text{in } [-1, 1]$$

we have

$$\begin{aligned} T_1 &\leq C_4 n \|Q_1 \Psi_n^j R_n^k w\|_{L_p[-1, 1]} \\ &\leq C_5 n \|Q_1 \Psi_n^j R_n^k w\|_{L_p[-a_{3n}, a_{3n}]} \\ &\leq C_6 n \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-a_{3n}, a_{3n}]} \end{aligned}$$

by first (2.10) and then (2.17). Next,

$$|\Psi'_n(x)| = 4 \left| 1 - \left(\frac{x}{a_n} \right)^2 \right| \frac{|x|}{a_n^2} \leq \frac{4}{a_n^2} \Psi_n(x)^{1/2}.$$

Using (2.17), we see that

$$T_2 \leq C_7 l \|P_n^{(l)} w \Psi_n^{l/4-1/4}\|_{L_p[-a_{3n}, a_{3n}]} \leq C_8 l T(a_n)^{1/2} \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-a_{3n}, a_{3n}]}$$

as $\Psi_n^{-1/4}(x) \leq T(a_n)^{1/2}$. Similarly, using (2.18) and then (2.17),

$$\begin{aligned} T_3 &\leq C_9 \|Q_1 \Psi_n^j R_n^k \Psi_n^{-1/2} w \Psi_n^{1/4}\|_{L_p[-a_{s(l+n)}, a_{s(l+n)}]} \\ &\leq C_{10} \|P_n^{(l)} w \Psi_n^{l/4-1/4}\|_{L_p[-a_{3n}, a_{3n}]} \end{aligned}$$

as before. So T_3 admits the same estimate as T_2 . Substituting into (2.19) gives (2.13) and then (2.14) follows as $T(a_n)^{1/2} = O(n)$. ■

Finally, we present an estimate of differences:

LEMMA 2.4. *Let $0 < \delta < 1$; $L, M > 0$; $0 < p \leq \infty$.*

(a) *Let $s \in (0, 1]$ and $[a, b]$ be contained in one of the ranges*

$$|x| \leq a_{1/t} \left(1 - \left[\frac{s}{2\delta a_{1/t}} \right]^2 \right) \quad (2.20)$$

or

$$1 > |x| \geq a_{1/t} \left(1 + \left[\frac{s}{2\delta a_{1/t}} \right]^2 \right). \quad (2.21)$$

Then

$$\int_a^b |f(x \pm s\Phi_t(x))| dx \leq 2(1-\delta)^{-1} \int_{\bar{a}}^{\bar{b}} |f(x)| dx, \quad (2.22)$$

where

$$\left\{ \begin{matrix} \bar{a} \\ \bar{b} \end{matrix} \right\} := \left\{ \begin{matrix} \inf \\ \sup \end{matrix} \right\} \{x \pm s\Phi_t(x) : x \in [a, b]\}. \quad (2.23)$$

(b) *Let $r \geq 1$, $t \in (0, 1/M)$, $h \in (0, Mt)$, and $[a, b]$ be contained in one of the ranges (2.20), (2.21) with $s = Mrt$. Define \bar{a} and \bar{b} by (2.23) with $s = Mrt$. Assume, moreover, that*

$$[a, b] \subset [-a_{L/t}, a_{L/t}]. \quad (2.24)$$

Then

$$\begin{aligned} & \|w(x) \mathcal{A}_{h\Phi_t(x)}^r(g, x, (-1, 1))\|_{L_p[a, b]} \\ & \leq C \inf_{P \in \mathcal{P}_{r-1}} \|w(g - P)\|_{L_p[\bar{a}, \bar{b}]} \leq C \|wg\|_{L_p[\bar{a}, \bar{b}]}. \end{aligned} \quad (2.25)$$

Here $C \neq C(a, b, t, h, g)$.

Proof. (a) Define $\sigma = \pm 1$ and $u(x) := x + \sigma s \Phi_t(x)$. We shall assume that $[a, b]$ is contained in the range (2.20) and also $a \geq 0$. The case where $a < 0$ is similar, as is the case when $[a, b]$ is contained in the range (2.21). Then for $x \in [a, b]$,

$$u'(x) = 1 + \frac{\sigma s}{2\sqrt{1 - (x/a_{1/t})}} \left(-\frac{1}{a_{1/t}} \right) \geq 1 - \delta$$

by (2.20). Hence u is increasing in $[a, b]$, and writing $v := u(x)$,

$$\begin{aligned} & \int_a^b |f(x \pm s\Phi_t(x))| dx \\ & = \int_a^b |f(u(x))| dx \\ & \leq \frac{1}{1 - \delta} \int_a^b |f(u(x))| u'(x) dx = (1 - \delta)^{-1} \int_{u(a)}^{u(b)} |f(v)| dv. \end{aligned}$$

So we have (2.22). The extra 2 in (2.22) takes care of having to split $[a, b]$ into two intervals if $a < 0 < b$.

(b) We shall assume that $p < \infty$. The proof is easier for $p = \infty$. Let us assume that $[a, b]$ is contained in the range (2.20) with $s = Mrt$. Now

$$w(x) \mathcal{A}_{h\Phi_t(x)}^r(g, x, (-1, 1)) = \sum_{i=0}^r \binom{r}{i} (-1)^i w(x) g \left(x + \left(\frac{r}{2} - i \right) h\Phi_t(x) \right).$$

Next, (2.8) shows that

$$w(x) \sim w \left(x \pm \left(\frac{r}{2} - i \right) h\Phi_t(x) \right)$$

uniformly in i and for $|x| \leq a_{1/t}$ and $h \leq Mt$. (It is only here that we need (2.24).) Then using (a), we obtain

$$\begin{aligned} \|w(x) \Delta'_{h\Phi_t(x)}(g, x, (-1, 1))\|_{L_p[a, b]}^p &\leq C \sup_{0 \leq i \leq r} \int_a^b |gw|^p \left(x \pm \frac{ih}{2} \Phi_t(x) \right) dx \\ &\leq \frac{2C}{1-\delta} \int_{\bar{a}}^{\bar{b}} |gw|^p(x) dx. \end{aligned}$$

Note that for $0 \leq i \leq r$, and $h \leq Mt$, our hypothesis (2.20) with $s = Mrt$ gives

$$|x| \leq a_{1/t} \left(1 - \left(\frac{Mrt}{2\delta a_{1/t}} \right)^2 \right) \leq a_{1/t} \left(1 - \left(\frac{ih}{2\delta a_{1/t}} \right)^2 \right)$$

so the range restrictions of (a) are satisfied. Finally note that for $P \in \mathcal{P}_{r-1}$,

$$\Delta'_{h\Phi_t(x)}(P, x, (-1, 1)) \equiv 0.$$

Hence

$$\begin{aligned} &\|w(x) \Delta'_{h\Phi_t(x)}(g, x, (-1, 1))\|_{L_p[a, b]} \\ &= \|w(x) \Delta'_{h\Phi_t(x)}(g - P, x, (-1, 1))\|_{L_p[a, b]} \leq C \|w(g - P)\|_{L_p[\bar{a}, \bar{b}]}. \end{aligned}$$

Now take inf's over P . ■

3. A CRUCIAL INEQUALITY

In this section, we establish a crucial inequality using ideas of Z. Ditzian [8].

THEOREM 3.1. *Let $w \in \mathcal{E}$. Let $r \geq 1$, $L > 0$, $0 < p \leq \infty$, $P_n \in \mathcal{P}_n$. If $0 < p < 1$, assume also the Markov–Bernstein inequality (1.16). Define $P \in \mathcal{P}_{r-1}$ by*

$$P(x) := P_n(x) - \int_{a_{Ln}}^x \int_{a_{Ln}}^{u_{r-1}} \cdots \int_{a_{Ln}}^{u_1} P_n^{(r)}(u_0) du_0 du_1 \cdots du_{r-1}. \quad (3.1)$$

Then for some $C_j \neq C_j(n, P)$, $j = 1, 2$,

$$\|(P_n - P)w\|_{L_p[a_{Ln}, 1]} \leq C_1(nT(a_n)^{1/2})^{-r} \|P_n^{(r)}w\|_{L_p[-1, 1]} \quad (3.2)$$

$$\leq C_2 n^{-r} \|P_n^{(r)}w\Phi_{1/n}^r\|_{L_p[-1, 1]}. \quad (3.3)$$

Our method of proof follows that of [8, 3]. The chief difficulty lies in the case $p < 1$. We first deal with $p \geq 1$, following the approach of [8].

LEMMA 3.2. Assume the hypotheses of Theorem 3.1 with $p \geq 1$. Then $\forall g \in L_p[a_{Ln}, 1]$,

$$\left\| w(x) \int_{a_{Ln}}^x g(u) du \right\|_{L_p[a_{Ln}, 1]} \leq C_1 (nT(a_n)^{1/2})^{-1} \|gw\|_{L_p[a_{Ln}, 1]}. \quad (3.4)$$

Here $C_1 \neq C_1(n, g)$.

Proof. This is similar to Lemma 11.4.1 in [9, p. 187], to Lemma 6.2 in [8], and to Lemma 4.2 in [3], but we provide the details. We begin by noting that for $x \geq t > 0$,

$$w(x)^{1/2} \int_t^x w(u)^{-1/2} Q'(u) du = 2 \left(1 - \left(\frac{w(x)}{w(t)} \right)^{1/2} \right) \leq 2. \quad (3.5)$$

Next for $n \geq n_0$ and $u \geq a_{Ln}$,

$$Q'(u) \geq C_1 Q'(a_{Ln}) \geq C_2 nT(a_n)^{1/2}. \quad (3.6)$$

(See (2.1).) Consequently for $x \geq a_{Ln}$,

$$\begin{aligned} w(x) \left| \int_{a_{Ln}}^x g(u) du \right| &\leq [C_2 nT(a_n)^{1/2}]^{-1} w(x)^{1/2} \int_{a_{Ln}}^x |gw| \\ &\quad \times (u) Q'(u) w^{-1/2}(u) du. \end{aligned} \quad (3.7)$$

We shall need a consequence of Jensen's inequality for integrals, applied with the power function t^p , $p \geq 1$: For non-negative measures μ and μ -measurable functions f ,

$$\left| \int f d\mu \right|^p \leq \left(\int |f|^p d\mu \right) \left(\int d\mu \right)^{p-1}. \quad (3.8)$$

Now we turn to the proof of (3.4):

$p = \infty$: Here (3.7) implies that for $x \geq a_{Ln}$,

$$\begin{aligned} w(x) \left| \int_{a_{Ln}}^x g(u) du \right| &\leq [C_2 nT(a_n)^{1/2}]^{-1} \|gw\|_{L_\infty[a_{Ln}, 1]} \\ &\quad \times w(x)^{1/2} \int_{a_{Ln}}^x Q'(u) w^{-1/2}(u) du \\ &\leq 2 [C_2 nT(a_n)^{1/2}]^{-1} \|gw\|_{L_\infty[a_{Ln}, 1]}. \end{aligned}$$

Here we have used (3.5). So we obtain (3.4).

$p < \infty$: Here applying (3.7),

$$\begin{aligned}
& \left\| w(x) \int_{a_{Ln}}^x g(u) du \right\|_{L_p[a_{Ln}, 1]} \\
& \leq [C_2 n T(a_n)^{1/2}]^{-1} \left[\int_{a_{Ln}}^1 \left[w(x)^{1/2} \int_{a_{Ln}}^x |gw| \right. \right. \\
& \quad \left. \left. \times (u) Q'(u) w^{-1/2}(u) du \right]^p dx \right]^{1/p} \\
& \leq [C_2 n T(a_n)^{1/2}]^{-1} \left[\int_{a_{Ln}}^1 2^{p-1} w(x)^{1/2} \int_{a_{Ln}}^x |gw|^p \right. \\
& \quad \left. \times (u) Q'(u) w^{-1/2}(u) du dx \right]^{1/p}
\end{aligned}$$

by (3.8) with $d\mu(u) := w(x)^{1/2} Q'(u) w(u)^{-1/2} du$ on $[a_{Ln}, x]$ and as (3.5) shows that $\int d\mu \leq 2$. Now

$$\begin{aligned}
& \int_{a_{Ln}}^1 w(x)^{1/2} \int_{a_{Ln}}^x |gw|^p (u) Q'(u) w(u)^{-1/2} du dx \\
& = \int_{a_{Ln}}^1 |gw|^p (u) \left[\int_u^1 w(x)^{1/2} Q'(u) dx \right] w(u)^{-1/2} du \\
& \leq C_3 \int_{a_{Ln}}^1 |gw|^p (u) \left[\int_u^1 w(x)^{1/2} Q'(x) dx \right] w(u)^{-1/2} du \\
& = 2C_3 \|gw\|_{L_p[a_{Ln}, 1]}^p.
\end{aligned}$$

In the second last line, we used the quasi-monotonicity of Q' . So (3.4) follows. ■

Proof of (3.3) for $p \geq 1$. This follows by induction on n from Lemma 3.2: Firstly for $r = 1$, Lemma 3.2 applied to $g = P'_n$ gives

$$\left\| w(x) \int_{a_{Ln}}^x P'_n(u_0) du_0 \right\|_{L_p[a_{Ln}, 1]} \leq C_1 (n T(a_n)^{1/2})^{-1} \|P'_n w\|_{L_p[a_{Ln}, 1]}$$

and then the identity (3.1) gives (3.2). As $\Phi_{1/n}(x) \geq T(a_n)^{-1/2}$, (3.3) also follows. Next for $r = 2$, Lemma 3.2 applied to

$$g(u_1) := \int_{a_{Ln}}^{u_1} P''_n(u_0) du_0$$

gives

$$\begin{aligned}
& \left\| w(x) \int_{a_{Ln}}^x \int_{a_{Ln}}^{u_n} P_n''(u_0) du_0 du_1 \right\|_{L_p[a_{Ln}, 1]} \\
&= \left\| w(x) \int_{a_{Ln}}^x g(u_1) du_1 \right\|_{L_p[a_{Ln}, 1]} \\
&\leq C_2(nT(a_n)^{1/2})^{-1} \|gw\|_{L_p[a_{Ln}, 1]} \\
&= C_2(nT(a_n)^{1/2})^{-1} \left\| w(x) \int_{a_{Ln}}^x P_n''(u_0) du_0 \right\|_{L_p[a_{Ln}, 1]} \\
&\leq C_3(nT(a_n)^{1/2})^{-2} \|wP_n''\|_{L_p[a_{Ln}, 1]},
\end{aligned}$$

by Lemma 3.2 again. Then (3.1) gives the result. Clearly after applying Lemma 3.2 r times to the right-hand side of (3.1), we get the result. ■

We break down the proof of Theorem 3.1 for $p < 1$ into a number of lemmas.

LEMMA 3.3. *Let $0 < p < 1$, $n \geq r \geq 1$, $L > 0$. Let $P_n \in \mathcal{P}_n$, $S \in \mathcal{P}_{r-1}$ and let*

$$g(x) := (P_n - S)(x). \quad (3.9)$$

Let

$$I_n(x) := \left\| |g'w|^{1-p}(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} \right\|_{L_\infty[a_{Ln}, x]}^{p/(1-p)}, \quad x \geq a_{Ln}. \quad (3.10)$$

Then

$$\begin{aligned}
\int_{a_{Ln}}^1 I_n(x) dx &\leq C \left[\sum_{j=1}^{r-1} (nT(a_n)^{1/2})^{-(j-1)p} \|w(P_n - S)^{(j)}\|_{L_p[a_{Ln}, 1]}^p \right. \\
&\quad \left. + (nT(a_n)^{1/2})^{-(r-1)p} \|wP_n^{(r)}\|_{L_p[-1, 1]}^p \right]. \quad (3.11)
\end{aligned}$$

Here C is independent of n , P_n , S .

Proof. Note first that

$$I_n(x) = \left\| |g'w|^p(u) \left(\frac{w(x)}{w(u)} \right)^{p/(2(1-p))} \right\|_{L_\infty[a_{Ln}, x]}.$$

Let $\tau := \delta(nT(a_n)^{1/2})^{-1}$, where $\delta > 0$ is so small that for $n \geq 1$, $R \in \mathcal{P}_n$,

$$T(a_n)^{-1/2} \|R'w\|_{L_p[-1, 1]} \leq \|R'w\Phi_{1/n}\|_{L_p[-1, 1]} \leq \frac{n}{2\delta} \|Rw\|_{L_p[-1, 1]}. \quad (3.12)$$

Given $x \geq a_{Ln}$, we define

$$k_0 := k_0(x) := \max\{k: x - (k+1)\tau \geq a_{Ln}\}.$$

We see that

$$\begin{aligned} I_n(x) &\leq \max_{0 \leq k \leq k_0(x)} \left\| |g'w|^p(u) \left(\frac{w(x)}{w(u)} \right)^{p/(2(1-p))} \right\|_{L_\infty[x - (k+1)\tau, x - k\tau]} \\ &\quad + \left\| |g'w|^p(u) \left(\frac{w(x)}{w(u)} \right)^{p/(2(1-p))} \right\|_{L_\infty[a_{Ln}, x - (k_0+1)\tau]}. \end{aligned}$$

Now for $u \in [x - (k+1)\tau, x - k\tau]$, (recall that $x - (k+1)\tau \geq a_{Ln}$)

$$\frac{w(x)}{w(u)} \leq e^{Q(x-k\tau) - Q(x)}$$

and by (3.6),

$$Q(x) - Q(x - k\tau) \geq C_2 n T(a_n)^{1/2} k\tau = C_2 \delta k. \quad (3.13)$$

Thus

$$\left(\frac{w(x)}{w(u)} \right)^{p/(2(1-p))} \leq q^k, \quad u \in [x - (k+1)\tau, x - k\tau],$$

where $q \in (0, 1)$ is independent of x, u, k . Then

$$\begin{aligned} I_n(x) &\leq \max_{0 \leq k \leq k_0(x)} q^k \|g'w\|_{L_\infty[x - (k+1)\tau, x - k\tau]}^p + q^{k_0(x)} \|g'w\|_{L_\infty[a_{Ln}, x - (k_0+1)\tau]}^p \\ &\leq \sum_{k=0}^{k_0(x)} q^k \|g'w\|_{L_\infty[x - (k+1)\tau, x - k\tau]}^p + q^{k_0(x)} \|g'w\|_{L_\infty[a_{Ln}, x - (k_0+1)\tau]}^p. \end{aligned}$$

Let us set $I_n(x) := 0 = (g'w)(x)$, $x > 1$, so that the previous inequality remains valid for $x \geq 1$. This device simplifies the subsequent argument. Then

$$\begin{aligned}
\int_{a_{L_n}}^1 I_n(x) dx &= \sum_{m=0}^{\infty} \int_{a_{L_n}+m\tau}^{a_{L_n}+(m+1)\tau} I_n(x) dx \\
&\leq \sum_{m=0}^{\infty} \int_{a_{L_n}+m\tau}^{a_{L_n}+(m+1)\tau} \left[\sum_{k=0}^{k_0(x)} q^k \|g'w\|_{L_{\infty}[x-(k+1)\tau, x-k\tau]}^p \right. \\
&\quad \left. + q^{k_0(x)} \|g'w\|_{L_{\infty}[a_{L_n}, x-(k_0+1)\tau]}^p \right] dx.
\end{aligned}$$

We observe that

$$\int_{a_{L_n}+m\tau}^{a_{L_n}+(m+1)\tau} \|g'w\|_{L_{\infty}[x-(k+1)\tau, x-k\tau]}^p dx = \int_{a_{L_n}+(m-k-1)\tau}^{a_{L_n}+(m-k)\tau} \|g'w\|_{L_{\infty}[x, x+\tau]}^p dx.$$

Recall that if $x \in [a_{L_n}+m\tau, a_{L_n}+(m+1)\tau]$, then $m \geq k_0 \geq m-1$, so

$$\begin{aligned}
&\int_{a_{L_n}}^1 I_n(x) dx \\
&\leq \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m-1} \int_{a_{L_n}+(m-k-1)\tau}^{a_{L_n}+(m-k)\tau} q^k \|g'w\|_{L_{\infty}[x, x+\tau]}^p dx \right. \\
&\quad \left. + 2q^{m-1} \int_{a_{L_n}}^{a_{L_n}+\tau} \|g'w\|_{L_{\infty}[a_{L_n}, x]}^p dx \right] \\
&\leq \sum_{s=0}^{\infty} \int_{a_{L_n}+s\tau}^{a_{L_n}+(s+1)\tau} \|g'w\|_{L_{\infty}[x, x+\tau]}^p dx \sum_{\substack{(m,k) \\ s=m-k-1}} q^k \\
&\quad + 2 \int_{a_{L_n}}^{a_{L_n}+\tau} \|g'w\|_{L_{\infty}[a_{L_n}, x]}^p dx \frac{1}{q(1-q)} \\
&\leq C_3 \left[\sum_{s=0}^{\infty} \int_{a_{L_n}+s\tau}^{a_{L_n}+(s+1)\tau} \|g'w\|_{L_{\infty}[x, x+\tau]}^p dx + \int_{a_{L_n}}^{a_{L_n}+\tau} \|g'w\|_{L_{\infty}[a_{L_n}, x]}^p dx \right].
\end{aligned} \tag{3.14}$$

Now for $u \in [x, x+\tau]$ (recall that $p < 1$)

$$\begin{aligned}
|g'(u)|^p &= \left| \sum_{j=1}^n \frac{g^{(j)}(x)}{(j-1)!} (u-x)^{j-1} \right|^p \\
&\leq \sum_{j=1}^n |g^{(j)}(x)|^p \tau^{(j-1)p} \\
&= \sum_{j=1}^{r-1} |(P_n - S)^{(j)}(x)|^p \tau^{(j-1)p} + \sum_{j=r}^n |P_n^{(j)}(x)|^p \tau^{(j-1)p}.
\end{aligned}$$

Since also $w(u) \leq w(x)$ for $u \in [x, x + \tau]$, we obtain (recall our convention $g'(x) = 0$, $x > 1$)

$$\begin{aligned}
& \sum_{s=0}^{\infty} \int_{a_{Ln} + s\tau}^{a_{Ln} + (s+1)\tau} \|g'w\|_{L_{\infty}[x, x+\tau]}^p dx \\
& \leq \sum_{j=1}^{r-1} \tau^{(j-1)p} \|(P_n - S)^{(j)}w\|_{L_p[a_{Ln}, 1]}^p \\
& \quad + \tau^{(r-1)p} \sum_{j=r}^n \tau^{(j-r)p} \|P_n^{(j)}w\|_{L_p[a_{Ln}, 1]}^p \\
& \leq \sum_{j=1}^{r-1} \tau^{(j-1)p} \|(P_n - S)^{(j)}w\|_{L_p[a_{Ln}, 1]}^p \\
& \quad + \tau^{(r-1)p} \|P_n^{(r)}w\|_{L_p[-1, 1]}^p \sum_{j=r}^n \left(\frac{\tau}{2\delta} nT(a_n)^{1/2} \right)^{(j-r)p}
\end{aligned}$$

(by (3.12))

$$\begin{aligned}
& \leq \sum_{j=1}^{r-1} \left(\frac{\delta}{nT(a_n)^{1/2}} \right)^{(j-1)p} \|(P_n - S)^{(j)}w\|_{L_p[a_{Ln}, 1]}^p \\
& \quad + (1 - 2^{-p})^{-1} \left(\frac{\delta}{nT(a_n)^{1/2}} \right)^{(r-1)p} \|P_n^{(r)}w\|_{L_p[-1, 1]}^p. \tag{3.15}
\end{aligned}$$

Here we have used our choice $\tau = \delta(nT(a_n)^{1/2})^{-1}$. In much the same way, we can use Taylor series to estimate the second term in the right-hand side of (3.14). Together (3.14) and (3.15) then give the result. ■

LEMMA 3.4. *Let $0 < p < 1$, $n \geq r \geq 1$, $L > 0$. Let $P_n \in \mathcal{P}_n$, $S \in \mathcal{P}_{r-1}$ and let*

$$(P_n - S)(a_{Ln}) = 0. \tag{3.16}$$

Then for some C independent of n , P_n , S

$$\begin{aligned}
& \|w(P_n - S)\|_{L_p[a_{Ln}, 1]} \leq C(nT(a_n)^{1/2})^{-1} \|w(P'_n - S')\|_{L_p[a_{Ln}, 1]}^p \\
& \quad \times \left[\sum_{j=1}^{r-1} (nT(a_n)^{1/2})^{-(j-1)(1-p)} \|w(P_n - S)^{(j)}\|_{L_p[a_{Ln}, 1]}^{1-p} \right. \\
& \quad \left. + (nT(a_n)^{1/2})^{-(r-1)(1-p)} \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \right]. \tag{3.17}
\end{aligned}$$

Proof. Let g be given by (3.9). Note that $g(a_{Ln}) = 0$ by our hypothesis (3.16). In particular

$$g(x) = \int_{a_{Ln}}^x g'(u) du.$$

Then

$$\begin{aligned} \Delta &:= \|w(P_n - S)\|_{L_p[a_{Ln}, 1]} = \|wg\|_{L_p[a_{Ln}, 1]} \\ &= \left\{ \int_{a_{Ln}}^1 \left| \int_{a_{Ln}}^x (g'w)(u) \frac{w(x)}{w(u)} du \right|^p dx \right\}^{1/p} \\ &\leq \left\{ \int_{a_{Ln}}^1 \left\| |g'w|^{1-p}(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} \right\|_{L_\infty[a_{Ln}, x]}^p \right. \\ &\quad \left. \times \left\{ \int_{a_{Ln}}^x |g'w|^p(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} du \right\}^p dx \right\}^{1/p}. \end{aligned}$$

Now we apply Hölder's inequality with parameters $r = 1/(1-p)$, $\sigma = 1/p$, so that $1/\sigma + 1/r = 1$:

$$\begin{aligned} \Delta &\leq \left\{ \int_{a_{Ln}}^1 \left\| |g'w|^{1-p}(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} \right\|_{L_\infty[a_{Ln}, x]}^{p/(1-p)} dx \right\}^{(1-p)/p} \\ &\quad \times \left\{ \int_{a_{Ln}}^1 \int_{a_{Ln}}^x |g'w|^p(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} du dx \right\} =: T_1 \times T_2. \end{aligned} \quad (3.18)$$

We see from (3.10), (3.11) of Lemma 3.3 that

$$\begin{aligned} T_1 &= \left\{ \int_{a_{Ln}}^1 I_n(x) dx \right\}^{(1-p)/p} \\ &\leq C_1 \left[\sum_{j=1}^{r-1} (nT(a_n)^{1/2})^{-(j-1)(1-p)} \|w(P_n - S)^{(j)}\|_{L_p[a_{Ln}, 1]}^{1-p} \right. \\ &\quad \left. + (nT(a_n)^{1/2})^{-(r-1)(1-p)} \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \right]. \end{aligned} \quad (3.19)$$

Of course C_1 is not the same C as in (3.11), but is independent of n and P_n . Next, using (3.6),

$$\begin{aligned}
T_2 &= \int_{a_{L_n}}^1 |g'w|^p(u) \int_u^1 \left(\frac{w(x)}{w(u)} \right)^{1/2} dx du \\
&\leq C_3(nT(a_n)^{1/2})^{-1} \int_{a_{L_n}}^1 |g'w|^p(u) \left[w(u)^{-1/2} \int_u^1 w(x)^{1/2} Q'(x) dx \right] du \\
&= 2C_3(nT(a_n)^{1/2})^{-1} \int_{a_{L_n}}^1 |g'w|^p(u) du \\
&= 2C_3(nT(a_n)^{1/2})^{-1} \|w(P'_n - S')\|_{L_p[a_{L_n}, 1]}^p.
\end{aligned}$$

Combining this with (3.18) and (3.19), we obtain the result. \blacksquare

Proof of Theorem 3.1 for $p < 1$. Let $P_n \in \mathcal{P}_n$ and let the corresponding P be given by (3.1). We begin by noting an extension of (3.17): For $0 \leq l < r$,

$$\begin{aligned}
&\|w(P_n - P)^{(l)}\|_{L_p[a_{L_n}, 1]} \\
&\leq C(nT(a_n)^{1/2})^{-1} \|w(P_n - P)^{(l+1)}\|_{L_p[a_{L_n}, 1]}^p \\
&\quad \times \left[\sum_{j=l+1}^{r-1} (nT(a_n)^{1/2})^{-(j-l-1)(1-p)} \|w(P_n - P)^{(j)}\|_{L_p[a_{L_n}, 1]}^{1-p} \right. \\
&\quad \left. + (nT(a_n)^{1/2})^{-(r-l-1)(1-p)} \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \right]. \tag{3.20}
\end{aligned}$$

The case $l=0$ of (3.20) is just (3.17) with $S=P$. The case $l>0$ follows by applying Lemma 3.4 to the polynomial $(P_n - P)^{(l)} \in \mathcal{P}_{n-l}$ and with r in (3.17) replaced by $r-l$. (Note that $(P_n - P)^{(l)}(a_{L_n})=0$, $0 \leq l < r$, so the hypothesis (3.16) is fulfilled). We use (3.20) and backward induction to show that

$$\|w(P_n - P)^{(k)}\|_{L_p[a_{L_n}, 1]} \leq C(nT(a_n)^{1/2})^{k-r} \|wP_n^{(r)}\|_{L_p[-1, 1]}, \tag{3.21}$$

$k = r-1, r-2, \dots, 0$. Of course Theorem 3.1 is the case $k=0$ of (3.21).

$k = r-1$. Here (3.20) with $l = r-1$ gives

$$\begin{aligned}
&\|w(P_n - P)^{(r-1)}\|_{L_p[a_{L_n}, 1]} \\
&\leq C(nT(a_n)^{1/2})^{-1} \|w(P_n - P)^{(r)}\|_{L_p[a_{L_n}, 1]}^p \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \\
&= C(nT(a_n)^{1/2})^{-1} \|wP_n^{(r)}\|_{L_p[-1, 1]}.
\end{aligned}$$

Assume (3.21) true for $r-1, r-2, \dots, k+1$. We prove (3.21) for $l=k$ by substituting (3.21) for $l=r-1, r-2, \dots, k+1$ into (3.20) with $l=k$. We obtain

$$\begin{aligned}
& \|w(P_n - P)^{(k)}\|_{L_p[a_{L_n}, 1]} \\
& \leq C(nT(a_n)^{1/2})^{-1} [(nT(a_n)^{1/2})^{k+1-r} \|wP_n^{(r)}\|_{L_p[-1, 1]}]^p \\
& \quad \times \left[\sum_{j=k+1}^{r-1} (nT(a_n)^{1/2})^{-(j-k-1)(1-p) + (j-r)(1-p)} \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \right. \\
& \quad \left. + (nT(a_n)^{1/2})^{-(r-k-1)(1-p)} \|wP_n^{(r)}\|_{L_p[-1, 1]}^{1-p} \right] \\
& \leq C_1(nT(a_n)^{1/2})^{k-r} \|wP_n^{(r)}\|_{L_p[-1, 1]}.
\end{aligned}$$

So we have (3.21) for k . ■

4. PROOF OF THEOREM 1.3

In this section, we shall prove Theorem 1.3. We first prove two lemmas, which together give most of the proof. Recall first that for $t > 0$,

$$\bar{K}_{r,p}(f, w, t^r) := \inf_{\deg(P) \leq 1/t} \{ \|(f - P)w\|_{L_p[-1, 1]} + t^r \|P^{(r)}w\Phi_t^r\|_{L_p[-1, 1]} \};$$

$$\begin{aligned}
\omega_{r,p}(f, w, t) &= \sup_{0 < h \leq t} \|w\Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2t)})} \\
&\quad + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P)w\|_{L_p(1 \geq |x| \geq a_{1/(4t)})};
\end{aligned}$$

and

$$\begin{aligned}
\bar{\omega}_{r,p}(f, w, t) &= \left[\frac{1}{t} \int_0^t \|w\Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2t)})}^p dh \right]^{1/p} \\
&\quad + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P)w\|_{L_p(1 \geq |x| \geq a_{1/(4t)})}.
\end{aligned}$$

Throughout this section, we set $q := \min\{1, p\}$ and we assume the hypotheses of Theorem 1.3 (unless otherwise specified). We begin by estimating $\omega_{r,p}$ above in terms of $\bar{K}_{r,p}$.

LEMMA 4.1. *Let $L > 0$, $0 < p \leq \infty$, $r \geq 1$. Then for $t \in (0, C_1)$ we have*

$$\omega_{r,p}(f, w, Lt) \leq C_2 \bar{K}_{r,p}(f, w, t^r), \quad (4.1)$$

where $C_j \neq C_j(f, t)$, $j = 1, 2$.

Proof. Fix $t < 1$ and determine P_n of degree at most $n := \langle 1/t \rangle \leq 1/t$ such that

$$\|(f - P_n) w\|_{L_p[-1, 1]} + t^r \|P_n^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \leq 2 \bar{K}_{r,p}(f, w, t^r). \quad (4.2)$$

We shall show that for $0 < h \leq Lt$,

$$\|w \Delta_{h\Phi_{Lt}(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2Lt)})} \leq C_1 \bar{K}_{r,p}(f, w, t^r) \quad (4.3)$$

and

$$\inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p(1 \geq |x| \geq a_{1/(4Lt)})} \leq C_2 \bar{K}_{r,p}(f, w, t^r), \quad (4.4)$$

which by definition of $\omega_{r,p}$ implies (4.1).

Proof of (4.3). Now

$$\begin{aligned} & \|w \Delta_{h\Phi_{Lt}(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2Lt)})}^q \\ & \leq \|w \Delta_{h\Phi_{Lt}(x)}^r(f - P_n, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2Lt)})}^q \\ & \quad + \|w \Delta_{h\Phi_{Lt}(x)}^r(P_n, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2Lt)})}^q \\ & \leq C_3 \|(f - P_n) w\|_{L_p[-1, 1]}^q + \|w \Delta_{h\Phi_{Lt}(x)}^r(P_n, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2Lt)})}^q, \end{aligned} \quad (4.5)$$

by Lemma 2.4(b). Note that given any $A > 0$, (2.3), (2.2) and then (2.4) show that for small enough t and $|x| \leq a_{1/(2Lt)}$

$$1 - \frac{|x|}{a_{1/(Lt)}} \geq 1 - \frac{a_{1/(2Lt)}}{a_{1/(Lt)}} \geq \frac{C}{T(a_{1/t})} \geq At^2$$

so x lies in the range (2.20) with $s = Lr \cdot Lt$ and Lemma 2.4(b) is applicable. We now proceed to the (fairly complicated) estimation of the second term on the right-hand side of (4.5). Recall that $\Delta_h^r S \equiv 0$ for $S \in \mathcal{P}_{r-1}$. In particular this applies to

$$S(u) = \sum_{l=0}^{r-1} \frac{P_n^{(l)}(x)}{l!} (u - x)^l.$$

Then using Taylor series and this remark, we see that

$$\begin{aligned} \Delta_{h\Phi_{Lt}(x)}^r P_n(x) &= \sum_{k=0}^r \binom{r}{k} (-1)^k P_n \left(x + \left(\frac{r}{2} - k \right) h\Phi_{Lt}(x) \right) \\ &= \sum_{k=0}^r \binom{r}{k} (-1)^k \sum_{l=r}^n \frac{((r/2 - k) h\Phi_{Lt}(x))^l}{l!} P_n^{(l)}(x). \end{aligned} \quad (4.6)$$

Then we deduce that

$$\begin{aligned} &\|w\Delta_{h\Phi_{Lt}(x)}^r(P_n, x, (-1, 1))\|_{L_p[-1, 1]}^q \\ &\leq \sum_{k=0}^r \binom{r}{k}^q \sum_{l=r}^n \frac{([r/2] h)^{lq}}{l!^q} \|P_n^{(l)} w\Phi_{Lt}^l\|_{L_p[-1, 1]}^q. \end{aligned} \quad (4.7)$$

Now as $n = \langle 1/t \rangle$, we have by (2.5) that

$$\Phi_{1/n}(x) \sim \Phi_{Lt}(x) \sim \Phi_t(x) \sim \Psi_n^{1/4}(x), \quad x \in (-1, 1), \quad (4.8)$$

where the constants in \sim are independent of x , t , and $n = n(t) = \langle 1/t \rangle$. Recall that Ψ_n was defined by (2.9). By repeated application of our Markov–Bernstein inequality Lemma 2.3,

$$\begin{aligned} \|P_n^{(l)} w\Phi_{Lt}^l\|_{L_p[-1, 1]} &\leq C_1^l \|P_n^{(l)} w\Psi_n^{l/4}\|_{L_p[-1, 1]} \\ &\leq C_1^l \|P_n^{(r)} w\Psi_n^{r/4}\|_{L_p[-1, 1]} C_2^{l-r} \prod_{j=r}^{l-1} (n + jT(a_n)^{1/2}). \end{aligned}$$

Since $T(a_n)^{1/2} = o(n)$, we see that given $\varepsilon > 0$, we have for $n \geq n_0(\varepsilon)$, $r \leq l \leq n$,

$$\prod_{j=r}^{l-1} (n + jT(a_n)^{1/2}) \leq C_3 n^{l-r} [\varepsilon^{l-r} l! + 1]$$

where C_3 and n_0 do not depend on l , n (nor on h , L above). It is also important that C_1 , C_2 above are independent of ε . We deduce that for $n \geq n_0(\varepsilon)$, (or equivalently for small t), and $0 < h \leq Lt$,

$$\begin{aligned} &\|w\Delta_{h\Phi_{Lt}(x)}^r(P_n, x, (-1, 1))\|_{L_p[-1, 1]}^q \\ &\leq 2^r C_4 \|P_n^{(r)} w\Psi_n^{r/4}\|_{L_p[-1, 1]}^q \left(\frac{r}{2} h \right)^{rq} \sum_{l=r}^n \frac{([r/2] h C_1 C_2 n)^{(l-r)q}}{l!^q} [\varepsilon^{(l-r)q} l!^q + 1] \\ &\leq C_5 \|P_n^{(r)} w\Phi_t^r\|_{L_p[-1, 1]}^q t^{rq} \sum_{l=r}^{\infty} \left(\frac{r}{2} L C_1 C_2 \right)^{(l-r)q} \left[\varepsilon^{(l-r)q} + \frac{1}{l!^q} \right] \\ &\leq C_6 t^{rq} \|P_n^{(r)} w\Phi_t^r\|_{L_p[-1, 1]}^q \end{aligned}$$

as $nt \leq 1$ and provided ε is chosen so small that $[r/2] LC_1 C_2 \varepsilon < 1$. This is possible as ε did not depend on L, n, C_1, C_2 but can be made arbitrarily small for $n \geq n_0$. Then using (4.2) and (4.5), we obtain (4.3).

Proof of (4.4). Next, we recall Lemma 3.1 from [9], proved there for weights on the whole real line, but valid without change for $[-1, 1]$: For $\xi > 0$,

$$\inf_{S \in \mathcal{P}_{r-1}} \|(f - S) w\|_{L_p(1 \geq |x| \geq \xi)} \\ \leq 2^{4/q-3} \left[\inf_{S \in \mathcal{P}_{r-1}} \|(f - S) w\|_{L_p[\xi, 1]} + \inf_{S \in \mathcal{P}_{r-1}} \|(f - S) w\|_{L_p[-1, -\xi]} \right].$$

We shall apply this with $\xi = a_{1/(4Lt)}$ and use Theorem 3.1 to estimate $\inf_{S \in \mathcal{P}_{r-1}} \|(f - S) w\|_{L_p[\xi, 1]}$. The term on $[-1, -\xi]$ is handled similarly. For P_n determined by (4.2), choose $P \in \mathcal{P}_{r-1}$ by the identity (3.1) with L replaced by $1/(4L)$. Then as $a_{1/(4Lt)} \geq a_{n/(4L)}$,

$$\inf_{S \in \mathcal{P}_{r-1}} \|(f - S) w\|_{L_p[a_{1/(4Lt)}, 1]}^q \\ \leq \|(f - P) w\|_{L_p[a_{1/(4Lt)}, 1]}^q \\ \leq \|(f - P_n) w\|_{L_p[a_{1/(4Lt)}, 1]}^q + \|(P_n - P) w\|_{L_p[a_{n/(4L)}, 1]}^q \\ \leq [2\bar{K}_{r,p}(f, w, t^r)]^q + C_5(nT(a_n)^{1/2})^{-rq} \|P_n^{(r)} w\|_{L_p[-1, 1]}^q$$

(by (4.2) and (3.2))

$$\leq [2\bar{K}_{r,p}(f, w, t^r)]^q + C_5 t^{rq} \|P_n^{(r)} w \Phi_t^r\|_{L_p[-1, 1]}^q \leq C_6 \bar{K}_{r,p}(f, w, t^r)^q.$$

Here we have used $\frac{1}{2} \leq nt \leq 1$, (4.8), and then (4.2). ■

The converse direction is more difficult. We first prove:

LEMMA 4.2. *There exist C_j , $j=1, 2$, and $0 < \varepsilon_0 < 1$ such that if $0 < \lambda < \varepsilon_0$, and $0 < s, t \leq C_1$ with*

$$\lambda \leq \frac{s}{t} \leq \varepsilon_0 \tag{4.9}$$

we have

$$\bar{\omega}_{r,p}(f, w, s) \leq C_2 \bar{\omega}_{r,p}(f, w, t). \tag{4.10}$$

Here $C_j, j=1, 2$ and ε_0 do not depend on f, s, t (but depend on λ).

Proof. We do this for $p < \infty$; $p = \infty$ is much easier. We split

$$\begin{aligned}
 \bar{\omega}_{r,p}(f, w, s)^p &\leq \frac{2^p}{s} \int_0^s [\|w \Delta_{h\Phi_s(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(3t)}}^p \\
 &\quad + \|w \Delta_{h\Phi_s(x)}^r(f, x, (-1, 1))\|_{L_p(a_{1/(3t)} \leq |x| \leq a_{1/(2s)})}^p] dh \\
 &\quad + 2^p \inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p(a_{1/(4s)} \leq |x| \leq 1)}^p \\
 &=: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.
 \end{aligned} \tag{4.11}$$

First, as $a_{1/(4s)} \geq a_{1/(4t)}$,

$$\mathcal{T}_3 \leq 2^p \inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p(a_{1/(4t)} \leq |x| \leq 1)}^p \leq 2^p \bar{\omega}_{r,p}^p(f, w, t). \tag{4.12}$$

Next, by Lemma 2.4(b),

$$\mathcal{T}_2 \leq 2^p \inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p(a_{1/(4t)} \leq |x| \leq 1)}^p \leq 2^p \bar{\omega}_{r,p}^p(f, w, t). \tag{4.13}$$

In applying that lemma, we note that our range of integration is of the form (2.20). Moreover, in working out \bar{a} , we used the fact that $\Phi_s(x)$ is a decreasing function of $x \in [0, a_{1/(2s)}]$, so

$$\begin{aligned}
 &\inf\{x - Mrs\Phi_s(x) : a_{1/(3t)} \leq x \leq a_{1/(2s)}\} \\
 &= a_{1/(3t)} - Mrs\Phi_s(a_{1/(3t)}) \\
 &\geq a_{1/(3t)} - MrCt\Phi_t(a_{1/(3t)})
 \end{aligned}$$

(by (2.6))

$$\geq a_{1/(3t)} + o(1/T(a_{1/t})) \geq a_{1/(4t)}$$

for small t , see (2.3) and (2.4). Note that the bound on how small t should be in no way depends on s or λ . It is more difficult to handle \mathcal{T}_1 . Let us divide $J := [-a_{1/(3t)}, a_{1/(3t)}]$ into $O(1/s)$ intervals J_k such that

$$|J_k| \leq s\Phi_s(x), \quad x \in J_k$$

for all k . Here $|J_k|$ denotes the length of J_k . Formally we may do this by choosing a large positive integer $n \sim 1/s$ and a partition

$$-a_{1/3t} = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = a_{1/3t}$$

with

$$\int_{\tau_k}^{\tau_{k+1}} \Phi_s^{-1}(x) dx \Big/ \int_{\tau_0}^{\tau_n} \Phi_s^{-1}(x) dx = \frac{1}{n}, \quad 0 \leq k \leq n$$

and we set $J_k := [\tau_k, \tau_{k+1}]$. Note that then by (2.8),

$$\Phi_s(x) \sim \Phi_s(y); w(x) \sim w(y), \quad x, y \in J_k \quad (4.14)$$

with constants independent of x, y, s, k . Then

$$\begin{aligned} \mathcal{T}_1 &= \frac{2^p}{s} \int_0^s \sum_k \int_{J_k} |w \Delta_{h\Phi_s(x)}^r(f, x, (-1, 1))|^p dx dh \\ &\leq C \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^s |\Delta_{h\Phi_s(x)}^r(f, x, (-1, 1))|^p dh dx \\ &= C \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^{s\Phi_s(x)/\Phi_t(x)} |\Delta_{u\Phi_t(x)}^r(f, x, (-1, 1))|^p \frac{\Phi_t(x)}{\Phi_s(x)} du dx. \end{aligned}$$

Now from (2.6), for some $C \neq C(s, t, \lambda)$,

$$\sup_{x \in (-1, 1)} \frac{s\Phi_s(x)}{t\Phi_t(x)} \leq C \frac{s}{t} \sqrt{\log \left(2 + \frac{s}{t} \right)} \leq 1$$

if $s/t \leq \varepsilon_0$. Here $\varepsilon_0 \neq \varepsilon_0(s, t, \lambda)$. Moreover, by (2.5), for $\lambda \leq s/t \leq \varepsilon_0$,

$$C_3 \leq \frac{\Phi_s(x)}{\Phi_t(x)} \leq C_4, \quad x \in (-1, 1).$$

Then

$$\begin{aligned} \mathcal{T}_1 &\leq C_5 \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^t |\Delta_{u\Phi_t(x)}^r(f, x, (-1, 1))|^p du dx \\ &\leq C_6 \frac{1}{t} \int_0^t \int_J |w \Delta_{u\Phi_t(x)}^r(f, x, (-1, 1))|^p dx du \\ &\leq C_6 \bar{\omega}_{r,p}(f, w, t)^p. \end{aligned}$$

In the last line we used (4.14). Together with (4.12) and (4.13), this gives the result. ■

LEMMA 4.3. *Let $0 < p \leq \infty, r \geq 1$. There exist $C_1, C_2, C_3 > 0$ independent of f and t such that for $0 < t < C_1$,*

$$\bar{K}_{r,p}(f, w, t^r) \leq C_2 \bar{\omega}_{r,p}(f, w, C_3 t). \quad (4.15)$$

Proof. Let $t > 0$ and $n := \langle 1/t \rangle$. By Theorem 1.2, for $n \geq C_3$ we can choose $P_n \in \mathcal{P}_n$ such that

$$\|(f - P_n) w\|_{L_p[-1, 1]} \leq C_7 \inf_{\rho \in [\frac{3}{4}, 1]} \bar{\omega}_{r, p} \left(f, w, C_8 \frac{\rho}{n} \right) \leq C_7 \bar{\omega}_{r, p}(f, w, C_8 t) \quad (4.16)$$

for $t < C_1 < \frac{1}{4}$, since then we can choose

$$\rho := \rho(t) := nt = \langle 1/t \rangle \quad t \in [\frac{3}{4}, 1].$$

We shall show that

$$t^r \|P_n^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \leq C_9 \bar{\omega}_{r, p}(f, w, C_8 t). \quad (4.17)$$

Once we have done this it follows that

$$\begin{aligned} \bar{K}_{r, p}(f, w, t^r) &\leq \|(f - P_n) w\|_{L_p[-1, 1]} + t^r \|P_n^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \\ &\leq C_{10} \bar{\omega}_{r, p}(f, w, C_8 t). \end{aligned}$$

To prove (4.17), we set $s := \delta t$, where $\delta \leq \frac{1}{8}$ so that

$$\frac{1}{2s} \geq \frac{4}{t} \geq 2n, \quad t \leq t_0 \quad (4.18)$$

and δ is also so small that we can apply Lemma 4.2 (with t replaced by $C_8 t$) to deduce that

$$\bar{\omega}_{r, p}(f, w, s) \leq C_{11} \bar{\omega}_{r, p}(f, w, C_8 t). \quad (4.19)$$

Of course $\delta \neq \delta(f, t)$ but C_{11} depends on δ . We proceed to prove (4.17). Note first that applying (4.6) to the monomial x^r (with Lt replaced by s) gives

$$r!(h\Phi_s(x))^r = \Delta_{h\Phi_s(x)}^r x^r = \sum_{k=0}^r \binom{r}{k} (-1)^k \frac{((\lfloor r/2 \rfloor - k) h\Phi_s(x))^r}{r!} r!$$

so we can rewrite (4.6) as

$$\begin{aligned} \Delta_{h\Phi_s(x)}^r P_n(x) - (h\Phi_s(x))^r P_n^{(r)}(x) \\ = \sum_{k=0}^r \binom{r}{k} (-1)^k \sum_{l=r+1}^n \frac{((\lfloor r/2 \rfloor - k) h\Phi_s(x))^l}{l!} P_n^{(l)}(x). \end{aligned}$$

Moreover, as $s \sim t \sim 1/n$, we have (see (2.5))

$$\Phi_s(x) \sim \Phi_t(x) \sim \Phi_{1/n}(x) \sim \Psi_n^{1/4}(x), \quad x \in (-1, 1) \quad (4.20)$$

with the constants in \sim independent of n and t . Then

$$\begin{aligned}
 \Gamma &:= \|w[\Delta_{h\Phi_s(x)}^r P_n(x) - (h\Phi_s(x))^r P_n^{(r)}(x)]\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q \\
 &\leq \sum_{k=0}^r \binom{r}{k}^q \sum_{l=r+1}^n \frac{([r/2]h)^{lq}}{l!^q} \|P_n^{(l)} w\Phi_s^l\|_{L_p[-1, 1]}^q \\
 &\leq \sum_{k=0}^r \binom{r}{k}^q \sum_{l=r+1}^n \frac{(C_{12}[r/2]h)^{lq}}{l!^q} \|P_n^{(l)} w\Psi_n^{l/4}\|_{L_p[-1, 1]}^q \\
 &\leq C_{14} \|P_n^{(r)} w\Psi_n^{r/4}\|_{L_p[-1, 1]}^q \left(C_{12} \frac{r}{2} h\right)^{rq} \sum_{l=r+1}^n \frac{(C_{12} C_{13} n[r/2]h)^{(l-r)q}}{l!^q} l!^q
 \end{aligned}$$

by repeated applications of our Markov–Bernstein inequality (2.14). It is important here that C_{12} , C_{13} are independent of t , n , h , P_n , l . Now if $nh \leq \Delta$, where Δ is a fixed positive small enough number, Lemma 2.2 and (4.18) above allow us to continue this as

$$\Gamma \leq \frac{1}{2} h^{rq} \|P_n^{(r)} w\Phi_s^r\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q.$$

It is crucial that Δ is independent of t , h , n , P_n , l . We deduce that for $0 < h \leq \Delta/n$ and hence for $0 < h \leq \Delta t$, and hence for $0 < h \leq \Delta s$,

$$\begin{aligned}
 &\|w\Delta_{h\Phi_s(x)}^r P_n(x)\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q \\
 &\geq h^{rq} \|wP_n^{(r)} \Phi_s^r\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q \\
 &\quad - \|w[\Delta_{h\Phi_s(x)}^r P_n(x) - (h\Phi_s(x))^r P_n^{(r)}(x)]\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q \\
 &\geq \frac{1}{2} h^{rq} \|wP_n^{(r)} \Phi_s^r\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^q \geq C_{14} h^{rq} \|wP_n^{(r)} \Phi_s^r\|_{L_p[-1, 1]}^q.
 \end{aligned}$$

(In the last step, we again used $1/2s \geq 2n$ and Lemma 2.2.) Raising to the (p/q) th power, integrating for h from 0 to Δs , and using (4.20), we obtain

$$\frac{1}{\Delta s} \int_0^{\Delta s} \|w\Delta_{h\Phi_s(x)}^r P_n(x)\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^p dh \geq C_{15} s^{rp} \|P_n^{(r)} w\Phi_s^r\|_{L_p[-1, 1]}^p.$$

Assuming, as we can, that $\Delta < 1$, we obtain

$$\begin{aligned}
 &t^{rp} \|P_n^{(r)} w\Phi_t^r\|_{L_p[-1, 1]}^p \\
 &\leq \frac{C_{16}}{s} \int_0^s \|w\Delta_{h\Phi_s(x)}^r P_n(x)\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^p dh \\
 &\leq \frac{C_{17}}{s} \int_0^s \{ \|w\Delta_{h\Phi_s(x)}^r (P_n - f)(x)\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^p \\
 &\quad + \|w\Delta_{h\Phi_s(x)}^r f(x)\|_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}^p \} dh \\
 &\leq C_{18} \{ \|w(P_n - f)\|_{L_p[-1, 1]}^p + \bar{\omega}_{r,p}(f, w, s)^p \}
 \end{aligned}$$

(by (2.25))

$$\leq C_{19} \bar{\omega}_{r,p}(f, w, C_8 t)^p$$

by (4.16) and (4.19). So we have (4.17). ■

We can now turn to the

Proof of Theorem 1.3. From Lemmas 4.1 and 4.3, for any fixed $L > 0$, $0 < t < t_0$, we have

$$\begin{aligned} \bar{\omega}_{r,p}(f, w, Lt) &\leq \omega_{r,p}(f, w, Lt) \leq C_1 \bar{K}_{r,p}(f, w, t') \\ &\leq C_2 \bar{\omega}_{r,p}(f, w, C_3 t) \leq C_2 \omega_{r,p}(f, w, C_3 t). \end{aligned} \quad (4.21)$$

Here it is important that C_3 is independent of L, f, t . Fix $M > 0$ and choose $L = MC_3$ and set $s = C_3 t$ to deduce that

$$\omega_{r,p}(f, w, Ms) \leq C_2 \omega_{r,p}(f, w, s).$$

So we have (1.19). Similarly (4.21) gives

$$\bar{\omega}_{r,p}(f, w, Ms) \leq C_2 \bar{\omega}_{r,p}(f, w, s).$$

Then (4.21) gives

$$\omega_{r,p}(f, w, s) \sim \bar{\omega}_{r,p}(f, w, s) \sim \bar{K}_{r,p}(f, w, s^r)$$

with constants in \sim independent of f, s . ■

5. PROOF OF COROLLARIES 1.4, 1.5, and 1.6

First, Corollary 1.4 follows from (1.19) of Theorem 1.3 and Theorem 1.2. We turn to the

Proof of Corollary 1.5. Let $t \leq C < \frac{1}{2}$ and choose $n := \langle 1/t \rangle$ and $l := \langle \log_2(1/t) \rangle$, so that

$$\frac{1}{2t} \leq 2^l \leq \frac{1}{t}$$

and

$$\frac{1}{2} \leq n/2^l \leq 2.$$

For $j \geq 1$, let P_j^* be a best polynomial approximation of degree $\leq j$ to f , so that

$$\|(f - P_j^*) w\|_{L_p[-1, 1]} = E_j[f]_{w, p}.$$

Moreover, we set $P_{2^{-l}}^* := P_0^*$. Then by (1.19) of Theorem 1.3,

$$\begin{aligned} \omega_{r, p}(f, w, t)^q &\leq C_1 \omega_{r, p}(f, w, n^{-1})^q \leq C_2 \bar{K}_{r, p}(f, w, n^{-r})^q \\ &\leq C_3 \inf_{\deg(P) \leq n} \{ \|(f - P) w\|_{L_p[-1, 1]}^q \\ &\quad + n^{-rq} \|P^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]}^q \} \\ &\leq C_4 \{ \|(f - P_{2^{l-1}}^*) w\|_{L_p[-1, 1]}^q + n^{-rq} \|P_{2^{l-1}}^{*(r)} w \Phi_{2^{-l}}^r\|_{L_p[-1, 1]}^q \} \end{aligned}$$

(by (2.5))

$$\leq C_5 \{ E_{2^{l-1}}[f]_{w, p}^q + t^{rq} \sum_{k=0}^{l-1} \| \{ P_{2^k}^* - P_{2^{k-1}}^* \}^{(r)} w \Phi_{2^{-l}}^r \|_{L_p[-1, 1]}^q \}$$

(as $P_{2^{-1}}^{*(r)} \equiv 0$)

$$\leq C_6 \{ E_{2^{l-1}}[f]_{w, p}^q + t^{rq} \sum_{k=0}^{l-1} (\log 2^{l-k})^{rq/2} \| \{ P_{2^k}^* - P_{2^{k-1}}^* \}^{(r)} w \Phi_{2^{-k}}^r \|_{L_p[-1, 1]}^q \}$$

(by (2.6))

$$\leq C_7 \{ E_{2^{l-1}}[f]_{w, p}^q + t^{rq} \sum_{k=0}^{l-1} (l-k)^{rq/2} 2^{rkq} \| \{ P_{2^k}^* - P_{2^{k-1}}^* \} w \|_{L_p[-1, 1]}^q \}$$

(by (2.14): recall r is fixed)

$$\begin{aligned} &\leq C_8 \{ E_{2^{l-1}}[f]_{w, p}^q + t^{rq} \sum_{k=0}^{l-1} (l-k)^{rq/2} 2^{rkq} E_{2^{k-1}}[f]_{w, p}^q \} \\ &\leq C_9 t^{rq} \sum_{k=-1}^l (l-k+1)^{rq/2} 2^{rkq} E_{2^k}[f]_{w, p}^q. \quad \blacksquare \end{aligned}$$

We turn to the

Proof of Corollary 1.6(a). We shall separately show that for $0 < h \leq t$,

$$\|w \mathcal{A}_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(2t)})} \leq Ct^r \|f^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \quad (5.1)$$

and

$$\inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p(1 \geq |x| \geq a_{1/(4t)})} \leq Ct^r \|f^{(r)} w \Phi_t^r\|_{L_p[-1, 1]}. \quad (5.2)$$

The corollary follows immediately from these two inequalities. We do these for $1 \leq p < \infty$; the case $p = \infty$ is easier.

Proof of (5.1). First note that for $u > 0$,

$$\begin{aligned} & A'_u(f, x, (-1, 1)) \\ &= \left| \int_{-u/2}^{u/2} \int_{-u/2}^{u/2} \cdots \int_{-u/2}^{u/2} f^{(r)}(x + t_1 + t_2 + \cdots + t_r) dt_1 dt_2 \cdots dt_r \right| \\ &\leq u^{r-1} \int_{-ru/2}^{ru/2} |f^{(r)}(x + s)| ds. \end{aligned} \quad (5.3)$$

Now we claim that

$$\Phi_t(x + s) \sim \Phi_t(x); w(x + s) \sim w(x) \quad (5.4)$$

uniformly for

$$|x| \leq a_{1/(2t)}; |s| \leq \frac{rt}{2} \Phi_t(x).$$

This follows from Lemma 2.1(c) provided that for this range of x, s

$$|x + s| \leq a_{1/t}. \quad (5.5)$$

Now if $x \geq 0$,

$$|x + s| \leq x_0 := x + \frac{rt}{2} \left[\sqrt{1 - \frac{x}{a_{1/t}}} + T(a_{1/t})^{-1/2} \right].$$

Here by (2.3), (2.4),

$$\frac{rt}{2\sqrt{1 - (x/a_{1/t})}} \leq \frac{rt}{2\sqrt{1 - (a_{1/(2t)}/a_{1/t})}} \leq CtT(a_{1/t})^{1/2} = o(1), \quad t \rightarrow 0+.$$

So, by definition of x_0 and this last inequality,

$$\begin{aligned} 1 - \frac{x_0}{a_{1/t}} &\geq 1 - \frac{x}{a_{1/t}} - o(1) \left(1 - \frac{x}{a_{1/t}} \right) - \frac{rt}{2a_{1/t}} T(a_{1/t})^{-1/2} \\ &\geq \left(1 - \frac{a_{1/(2t)}}{a_{1/t}} \right) (1 - o(1)) - o(T(a_{1/t})^{-1}) \geq CT(a_{1/t})^{-1} > 0 \end{aligned}$$

by (2.4), (2.3). So we have (5.5) for small enough t . Then from that and then (5.3) and then (5.4),

$$\begin{aligned}
 & |w(x) \Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))| \\
 & \leq C_4 w(x) (h\Phi_t(x))^{r-1} \int_{-rh\Phi_t(x)/2}^{rh\Phi_t(x)/2} |f^{(r)}(x+s)| ds \\
 & \leq C_5 \frac{h^r}{h\Phi_t(x)} \int_{-rh\Phi_t(x)/2}^{rh\Phi_t(x)/2} |w\Phi_t^r f^{(r)}|(x+s) ds. \tag{5.6}
 \end{aligned}$$

Now for $p > 1$, the maximal function operator

$$M[g](x) := \sup_{u>0} \frac{1}{2u} \int_{-u}^u |g(x+s)| ds$$

is bounded from L_p to L_p , so

$$\begin{aligned}
 \|w\Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x| \leq a_{1/(4t)})} & \leq C_6 h^r \|M[w\Phi_t^r f^{(r)}]\|_{L_p[-1, 1]} \\
 & \leq C_7 h^r \|w\Phi_t^r f^{(r)}\|_{L_p[-1, 1]},
 \end{aligned}$$

that is, we have (5.1). In the case where $p=1$, we integrate (5.6), interchange integrals, and obtain (5.1) again, using also (2.8).

Proof of (5.2). Write $n := \langle 1/t \rangle$. Note that $1/(4t) \geq n/4$ so that

$$\inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p(1 \geq |x| \geq a_{1/(4t)})} \leq \inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p(1 \geq |x| \geq a_{n/4})}.$$

In fact it suffices to estimate

$$\inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p[a_{n/4}, 1]}$$

as a similar estimate holds for the range $[-1, -a_{n/4}]$; recall again from [8, Lemma 3.1] that

$$\begin{aligned}
 & \inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p[a_{n/4} \leq |x| \leq 1]} \\
 & \leq 2^{4/q-3} \left[\inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p[a_{n/4}, 1]} + \inf_{P \in \mathcal{P}_{r-1}} \|(f-P)w\|_{L_p[-1, -a_{n/4}]} \right].
 \end{aligned}$$

Now if $r = 1$, we can use Lemma 3.2 to deduce that (recall that $p \geq 1$ and that Lemma 3.2 did not require a Markov–Bernstein inequality)

$$\begin{aligned}
& \inf_{P \in \mathcal{P}_0} \|(f - P) w\|_{L_p[a_{n/4}, 1]} \\
& \leq \|(f(x) - f(a_{n/4})) w(x)\|_{L_p[a_{n/4}, 1]} \\
& = \left\| w(x) \int_{a_{n/4}}^x f'(u) du \right\|_{L_p[a_{n/4}, 1]} \leq C_1 [nT(a_n)]^{-1/2} \|f' w\|_{L_p[a_{n/4}, 1]} \\
& \leq C_2 n^{-1} \|f' w \Phi_t\|_{L_p[a_{n/4}, 1]}.
\end{aligned}$$

Next, as in induction hypothesis, assume that for $k = 1, 2, \dots, r - 1$, we have

$$\inf_{P \in \mathcal{P}_{k-1}} \|(g - P) w\|_{L_p[a_{n/4}, 1]} \leq C_3 n^{-k} \|g^{(k)} w \Phi_t^k\|_{L_p[a_{n/4}, 1]}, \quad (5.7)$$

where $C \neq C(g, n)$. (We have just proved this for $k = 1$.) Applying this with $k = r - 1$ to $g := f'$, we can choose $S \in \mathcal{P}_{r-2}$ such that

$$\|(f' - S) w\|_{L_p[a_{n/4}, 1]} \leq C_3 n^{-(r-1)} \|f^{(r)} w \Phi_t^{r-1}\|_{L_p[a_{n/4}, 1]}.$$

Set

$$P_1(x) := f(a_{n/4}) + \int_{a_{n/4}}^x S(u) du.$$

Then

$$\begin{aligned}
\inf_{P \in \mathcal{P}_{r-1}} \|(f - P) w\|_{L_p[a_{n/4}, 1]} & \leq \|(f - P_1) w\|_{L_p[a_{n/4}, 1]} \\
& \leq \left\| w(x) \int_{a_{n/4}}^x (f' - S)(u) du \right\|_{L_p[a_{n/4}, 1]} \\
& \leq C_4 [nT(a_n)]^{-1/2} \|(f' - S) w\|_{L_p[a_{n/4}, 1]}
\end{aligned}$$

(by Lemma 3.2)

$$\begin{aligned}
& \leq C_5 n^{-r} T(a_n)^{-1/2} \|f^{(r)} w \Phi_t^{r-1}\|_{L_p[a_{n/4}, 1]} \\
& \leq C_5 n^{-r} \|f^{(r)} w \Phi_t^r\|_{L_p[a_{n/4}, 1]}.
\end{aligned}$$

So we have (5.7) for r and hence also (5.2). ■

In proving Corollary 1.6(b), we need

LEMMA 5.1. *Let $L > 0$, $0 < p \leq \infty$. Suppose that $P_n \in \mathcal{P}_n$ satisfies*

$$\|(g - P_n) w\|_{L_p[-1, 1]} \leq L \omega_{r, p} \left(g, w, \frac{1}{n} \right). \quad (5.8)$$

Then

$$\|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \leq C n^r \omega_{r, p} \left(g, w, \frac{1}{n} \right), \quad (5.9)$$

where $C \neq C(g, n)$.

Proof. Choose P_n^* such that

$$\begin{aligned} & \|(g - P_n^*) w\|_{L_p[-1, 1]} + n^{-r} \|P_n^{*(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \\ & \leq 2\bar{K}_{r, p}(g, w, n^{-r}) \leq C_2 \omega_{r, p} \left(g, w, \frac{1}{n} \right). \end{aligned}$$

Then

$$\|(P_n - P_n^*) w\|_{L_p[-1, 1]} \leq C_3 \omega_{r, p} \left(g, w, \frac{1}{n} \right).$$

From the Markov–Bernstein inequality (2.14), we deduce that

$$\|(P_n - P_n^*)^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \leq C_4 n^r \omega_{r, p} \left(g, w, \frac{1}{n} \right).$$

Then

$$\begin{aligned} \|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} & \leq C_5 \{ \|P_n^{*(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \\ & \quad + \|(P_n - P_n^*)^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \} \\ & \leq C_6 n^r \omega_{r, p} \left(g, w, \frac{1}{n} \right). \quad \blacksquare \end{aligned}$$

Finally, we give the

Proof of Corollary 1.6(b). It is clear that

$$\bar{K}_{r, p}(f, w, t^r) \geq K_{r, p}(f, w, t^r).$$

To prove the converse inequality, choose g such that

$$\|(f - g) w\|_{L_p[-1, 1]} + t^r \|g^{(r)} w \Phi_t^r\|_{L_p[-1, 1]} \leq 2K_{r, p}(f, w, t^r).$$

Let $n := \langle 1/t \rangle$. By Corollary 1.4, we can choose $P_n \in \mathcal{P}_n$ such that

$$\|(g - P_n) w\|_{L_p[-1, 1]} \leq C_2 \omega_{r, p} \left(g, w, \frac{1}{n} \right).$$

Then by the lemma above,

$$n^{-r} \|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \leq C_2 \omega_{r, p} \left(g, w, \frac{1}{n} \right).$$

Then as $\Phi_t \sim \Phi_{1/n}$, we obtain (using (1.17), (1.19))

$$\begin{aligned} \bar{K}_{r, p}(f, w, t^r) &\leq C_3 \bar{K}_{r, p} \left(f, w, \frac{1}{n^r} \right) \\ &\leq C_3 \{ \|(f - P_n) w\|_{L_p[-1, 1]} + n^{-r} \|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \} \\ &\leq C_3 \{ \|(f - g) w\|_{L_p[-1, 1]} + \|(g - P_n) w\|_{L_p[-1, 1]} \\ &\quad + n^{-r} \|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \} \\ &\leq C_4 \left\{ K_{r, p}(f, w, t^r) + \omega_{r, p} \left(g, w, \frac{1}{n} \right) \right\} \\ &\leq C_5 \{ K_{r, p}(f, w, t^r) + n^{-r} \|g^{(r)} w \Phi_{1/n}^r\|_{L_p[-1, 1]} \} \\ &\leq C_6 K_{r, p}(f, w, t^r) \end{aligned}$$

by Corollary 1.6(a) and choice of g . ■

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